SOME NON-LINEAR PROBLEMS IN APPROXIMATION*

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1. **Introduction.** Much has been written on the least-square properties of developments in series of orthogonal functions. The approximating function, characterized by the requirement that it shall reduce the integral of the square of the error to a minimum, is chosen in each case from a linear family (for example, the family of all trigonometric sums of the nth order), containing the sum of any two of its members, or the product of any of them by a constant. The writer has considered the problem of least-square approximation, and more generally of approximation according to the criterion of least mth powers, in terms of the non-linear family of all functions $\phi(x)$ which satisfy the condition

$$|\phi(x_2) - \phi(x_1)| \leqq \lambda |x_2 - x_1|$$

for a given value of λ, and has discussed the convergence of the approximation as λ becomes infinite. Even in this case, the *average* of any two members of the family corresponding to a specified λ is contained in the family; and this fact enters into the proofs of uniqueness.†

In the problems of the present paper, an approximating function is to be chosen from a family which does not in general contain the average of two of its members. The cases taken as illustrative are those in which the approximating function is the square of a trigonometric sum of the nth order, or the square root of a (positive) trigonometric sum of the nth order, the function to be represented being itself positive. Questions of uniqueness, which would at any rate call for novel methods of treatment, are allowed to lapse. Even so, it is found possible to deal with the convergence of the approximations as the order of the sums becomes infinite. The method is essentially that which the writer has used repeatedly in connection with problems of approximation by the method of least mth powers. It involves, however, an extension of Bernstein's theorem, which will be obtained in the next section.

2. **Extension of Bernstein's theorem.** The familiar statement of Bernstein's theorem is that if $T_n(x)$ is a trigonometric sum of the nth order such

* Presented to the Society, December 28, 1926; received by the editors October 12, 1927.

† See D. Jackson, *On approximation by functions of given continuity*, these Transactions, vol. 25 (1923), pp. 449–458; p. 450.

that $|T_n(x)| \leqq L$ for all values of x, then $|T_n'(x)|$ can not exceed nL. The statement can be generalized as follows:

If $f(x)$ is a function of period 2π having a continuous first derivative subject everywhere to the condition $|f'(x)| \leqq \lambda$, and if $T_n(x)$ is a trigonometric sum of the nth order such that $|T_n(x) - f(x)| \leqq L$ for all values of x, then

$$|T_n'(x)| \leqq nL + C\lambda, \tag{1}$$

where C is an absolute constant. More definitely (though the preceding statement is sufficient for the applications)

$$|T_n'(x)| \leqq nL + 4\lambda.$$

This formulation reduces to the standard one if $f(x)$ is identically zero.

Since $f(x)$ satisfies a Lipschitz condition with coefficient λ, there exists a trigonometric sum $t_n(x)$, of the nth order, such that

$$|t_n(x) - f(x)| \leqq \frac{K\lambda}{n}, \tag{2}$$

where K is an absolute constant.* Such a sum is defined by the formula

$$t_n(x) = I_m(x) = \frac{1}{2}h_m \int_{-\pi}^{\pi} f(v)\Phi_m(v - x)dv, \tag{3}$$

where

$$\Phi_m(v) = \frac{\sin^4 (mv/2)}{m^4 \sin^4 (v/2)}, \quad \frac{2}{h_m} = \int_{-\pi}^{\pi} \Phi_m(v)dv = \int_{-\pi}^{\pi} \Phi_m(v - x)dv,$$

the value of the last expression being only apparently and not actually dependent on x. The conditions for differentiating under the sign of integration are satisfied in (3), so that

$$\begin{aligned} t_n'(x) &= \frac{1}{2}h_m \int_{-\pi}^{\pi} f(v)\frac{\partial}{\partial x}\Phi_m(v - x)dv \\ &= -\frac{1}{2}h_m \int_{-\pi}^{\pi} f(v)\frac{\partial}{\partial v}\Phi_m(v - x)dv \\ &= \frac{1}{2}h_m \int_{-\pi}^{\pi} f'(v)\Phi_m(v - x)dv, \end{aligned}$$

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* For the theorem, as well as for the details of proof referred to in the text, see D. Jackson, *On approximation by trigonometric sums and polynomials*, these Transactions, vol. 13 (1912), pp. 491–515; pp. 492–494.

the last form resulting from an integration by parts, with attention to the periodicity of the functions involved. Hence

$$(4) \qquad | t_n'(x) | \leqq \frac{1}{2} h_m \lambda \int_{-\pi}^{\pi} \Phi_m(v - x) dv = \lambda .$$

Let $\tau_n(x) = T_n(x) - t_n(x)$. Then

$$| \tau_n(x) | = | [T_n(x) - f(x)] - [t_n(x) - f(x)] | \leqq L + \frac{K\lambda}{n} \cdot$$

As $\tau_n(x)$ is a trigonometric sum, Bernstein's theorem is applicable, to the effect that

$$(5) \qquad | \tau_n'(x) | \leqq nL + K\lambda .$$

As $T_n'(x) = \tau_n'(x) + t_n'(x)$, it follows from (4) and (5) that

$$| T_n'(x) | \leqq nL + (K + 1)\lambda = nL + C\lambda, \quad C = K + 1 .$$

Since (2) is true* with $K = 3$, the conclusion (1) holds for $C = 4$.

An essential point for the applications is that C is independent of n. As a constant may be subtracted from $f(x)$ and from $T_n(x)$ without affecting the essential conditions of the problem, it may be assumed without loss of generality that $f(x)$ vanishes somewhere in a period, and then it is obvious (with the use of Bernstein's theorem) that

$$| f(x) | \leqq \pi\lambda, \quad | T_n(x) | \leqq L + \pi\lambda, \quad | T_n'(x) | \leqq n(L + \pi\lambda) .$$

But this comparatively trivial observation would not serve the purpose.

3. **Approximation by squares of trigonometric sums.** Let $f(x)$ be a given function of period 2π, which is positive everywhere and which has everywhere a continuous first derivative, and let λ be the maximum of $|f'(x)|$. Among all trigonometric sums of given order n, let $T_n(x)$ be one for which the integral

$$\int_{-\pi}^{\pi} \{f(x) - [T_n(x)]^2\}^2 dx$$

has the smallest possible value. It can be inferred readily from well known theorems that the greatest lower bound of this expression is a minimum which is actually attained. For if an upper bound is assigned to the value of the integral, the coefficients in $[T_n(x)]^2$, considered as a trigonometric sum of order $2n$, must belong to a bounded domain; the maximum of $[T_n(x)]^2$, and hence that of $|T_n(x)|$, is thereby restricted; and so the coefficients that

* These Transactions, vol. 13, loc. cit., Theorem VI, p. 510.

come into consideration for $T_n(x)$ belong to a bounded region, which may be taken as closed, so that the fundamental theorem on the existence of a minimum is applicable. On the other hand, the average of the squares of two trigonometric sums is not in general the square of a trigonometric sum,* and because of this circumstance the proof commonly given for the uniqueness of the minimum in similar cases breaks down. The determination of $T_n(x)$ is manifestly not unique, since $[-T_n(x)]$ would serve the same purpose; the question of the uniqueness of $[T_n(x)]^2$ will be left in abeyance. It will be supposed that a particular $T_n(x)$ minimizing the integral is designated for each value of n, and it will be shown that $[T_n(x)]^2$ *converges uniformly toward* $f(x)$ *as* n *becomes infinite.*

With the understanding, then, that $T_n(x)$ is a sum which minimizes the integral for a specified value of n, let

$$\gamma_n = \int_{-\pi}^{\pi} \{f(x) - [T_n(x)]^2\}^2 dx.$$

Let

$$R_n(x) = f(x) - [T_n(x)]^2,$$

let μ_n be the maximum of $|R_n(x)|$, and let x_0 be a value of x such that

$$|R_n(x_0)| = \mu_n.$$

As $[T_n(x)]^2$ is a trigonometric sum of order $2n$, and as $|f'(x)| \leqq \lambda$, it follows from the preceding section that

$$\left|\frac{d}{dx}[T_n(x)]^2\right| \leqq 2n\mu_n + C\lambda, \quad |R_n'(x)| \leqq 2n\mu_n + (C+1)\lambda.$$

Let it be supposed temporarily that $(C+1)\lambda \leqq n\mu_n$, the contrary case being reserved for later consideration. Then

$$|R_n'(x)| \leqq 3n\mu_n, \quad |R_n(x) - R_n(x_0)| \leqq 3n\mu_n |x - x_0|,$$

and for $|x - x_0| \leqq 1/(6n)$,

$$|R_n(x) - R_n(x_0)| \leqq \tfrac{1}{2}\mu_n, \qquad |R_n(x)| \geqq \tfrac{1}{2}\mu_n.$$

Since the last relation holds throughout an interval of length $1/(3n)$ at least,

$$\gamma_n \geqq (\tfrac{1}{2}\mu_n)^2/(3n).$$

On the other hand, since $f(x)$ has a positive minimum and possesses a continuous derivative, $[f(x)]^{1/2}$ also has a continuous derivative, and so

* E.g., the average of $2\sin^2 x$ and $4\cos^2 x$ is $1+\cos^2 x$, and it is readily seen that no trigonometric sum can have the last expression for its square.

satisfies a Lipschitz condition, with some coefficient which may be denoted by λ'. Therefore a trigonometric sum $t_n(x)$ of the nth order can be constructed so that

$$| [f(x)]^{1/2} - t_n(x) | \leqq K\lambda'/n,$$

where K is the absolute constant already cited. If such sums are formed for all positive integral values of n, they are uniformly bounded; the expressions

$$| [f(x)]^{1/2} + t_n(x) |$$

have an upper bound A which is independent of n, and

$$| f(x) - [t_n(x)]^2 | \leqq AK\lambda'/n.$$

By reason of the minimizing property of $T_n(x)$,

$$\gamma_n \leqq \int_{-\pi}^{\pi} \{f(x) - [t_n(x)]^2\}^2 dx \leqq 2\pi(AK\lambda'/n)^2.$$

If $2\pi(AK\lambda')^2 = B$, then,

$$(\tfrac{1}{2}\mu_n)^2/(3n) \leqq \gamma_n \leqq B/n^2, \qquad \mu_n \leqq 2(3B)^{1/2}/n^{1/2}.$$

These relations have been deduced on the hypothesis that $(C+1)\lambda \leqq n\mu_n$. To deny this hypothesis, however, is to suppose directly that $\mu_n < (C+1)\lambda/n$. If D is the larger of the numbers $2(3B)^{1/2}$, $(C+1)\lambda$, it is certain in any case that

$$\mu_n \leqq D/n^{1/2},$$

which means that $\lim_{n=\infty} \mu_n = 0$, and the convergence is proved.

4. Approximation by square roots of trigonometric sums. Let $f(x)$ be subject to the same hypotheses as before. Let $T_n(x)$ be characterized this time, however, among all non-negative trigonometric sums of the nth order, by the requirement that the integral

$$\int_{-\pi}^{\pi} \{f(x) - [T_n(x)]^{1/2}\}^2 dx$$

shall be a minimum. In the preceding section, the approximating function belonged to a certain restricted class of trigonometric sums; here it is not in general a trigonometric sum at all.

Informally stated, the argument for the existence of a minimum is as follows. Consider the integral

$$\int_{-\pi}^{\pi} \{f(x) - [t_n(x)]^{1/2}\}^2 dx$$

formed with an arbitrary trigonometric sum $t_n(x)$ of the nth order. If any coefficient in $t_n(x)$ were very large, the value of the expression in braces would be large at some point. The magnitude of its derivative being restricted (somewhat indirectly, but in fairly obvious fashion) by Bernstein's theorem, the expression itself would necessarily remain large over an appreciable interval, and the integral would be large. The approach of the integral to its greatest lower bound can take place therefore only in a restricted domain for the coefficients, where the fundamental theorem on the existence of a minimum is once more in force.

The method that first comes to mind in connection with an attempt to establish the uniqueness of the minimum is blocked again, by the fact this time that the average of the square roots of two trigonometric sums is not in general the square root of a trigonometric sum.* The question of uniqueness will accordingly be dismissed from further consideration, and it will be supposed merely that $T_n(x)$ denotes for each positive integral n a particular sum of the nth order for which the integral has its minimum value, whether it be the only sum having this property or not.

With this understanding, let

$$\gamma_n = \int_{-\pi}^{\pi} \{f(x) - [T_n(x)]^{1/2}\}^2 dx.$$

The problem is to show that $[T_n(x)]^{1/2}$ converges uniformly toward $f(x)$. Because of the fact that $[T_n(x)]^{1/2}$ is not a trigonometric sum, several preliminary steps are needed before the lemma of §2 can be applied. Let λ' be the maximum of $|2f(x)f'(x)|$. Then $[f(x)]^2$ satisfies a Lipschitz condition with coefficient λ', and there exists a trigonometric sum $t_n(x)$, of the nth order, such that

$$|\, [f(x)]^2 - t_n(x)\,| \leqq K\lambda'/n. \tag{6}$$

As $[f(x)]^2$ has a positive minimum, it is clear from (6) itself that $t_n(x)$ is everywhere positive, at least for values of n from a certain point on; if it is constructed according to the procedure used in establishing the general theorem to which reference is made (cf. §2) it is in fact never less than the minimum of $[f(x)]^2$. Let a be the minimum of $f(x)$, a positive number, by hypothesis. Then

$$f(x) + [t_n(x)]^{1/2} \geqq a,$$

* For example, the average of $2(2^{1/2}+\sin x)^{1/2}$ and $2(2^{1/2}-\sin x)^{1/2}$ is the square root of

$$2^{3/2}+2(2-\sin^2 x)^{1/2} = 2^{3/2}+2(1+\cos^2 x)^{1/2},$$

and $(1+\cos^2 x)^{1/2}$, as already remarked, is not a trigonometric sum.

the second term representing naturally the positive square root, and

$$|f(x) - [t_n(x)]^{1/2}| \leqq K\lambda'/(an).$$

Consequently

$$\gamma_n \leqq \int_{-\pi}^{\pi} \{f(x) - [t_n(x)]^{1/2}\}^2 dx \leqq 2\pi(K\lambda')^2/(an)^2. \tag{7}$$

Let M_n be the maximum of $[T_n(x)]^{1/2}$, and let M be the maximum of $f(x)$. In consideration of the alternatives $M_n < 2M$, $M_n \geqq 2M$, let it be supposed for the sake of argument that the latter relation is fulfilled. Let x_1 be a value of x for which $[T_n(x_1)]^{1/2} = M_n$. If $[T_n(x)]^{1/2}$ is not everywhere greater than $\frac{3}{4}M_n$, let x_2 be the value of x nearest to x_1 for which $[T_n(x_2)]^{1/2} = \frac{3}{4}M_n$, and let $|x_2 - x_1| = \delta$. Then

$$\left|\frac{d}{dx}[T_n(x)]^{1/2}\right| = \frac{|T_n'(x)|}{2[T_n(x)]^{1/2}} \leqq \frac{2|T_n'(x)|}{3M_n} \leqq \frac{|T_n'(x)|}{M_n},$$

for $|x - x_1| \leqq \delta$, since $[T_n(x)]^{1/2} \geqq \frac{3}{4}M_n$ for the values of x in question. But $|T_n(x)| \leqq M_n^2$, and hence $|T_n'(x)| \leqq nM_n^2$, by Bernstein's theorem. So

$$\left|\frac{d}{dx}[T_n(x)]^{1/2}\right| \leqq nM_n$$

for $|x - x_1| \leqq \delta$. This implies further that

$$\tfrac{1}{4}M_n = [T_n(x_1)]^{1/2} - [T_n(x_2)]^{1/2} \leqq nM_n|x_2 - x_1| = nM_n\delta,$$

$$\delta \geqq 1/(4n).$$

But as long as $|x - x_1| \leqq \delta$,

$$|f(x) - [T_n(x)]^{1/2}| \geqq \tfrac{3}{4}M_n - M \geqq \tfrac{1}{4}M_n, \tag{8}$$

since it is supposed for the time being that $M \leqq \frac{1}{2}M_n$, and consequently

$$\gamma_n \geqq 2\delta(\tfrac{1}{4}M_n)^2 \geqq 2\delta(\tfrac{1}{2}M)^2 \geqq M^2/(8n),$$

which is in contradiction with (7) for all values of n from a certain point on. To go back to one of the alternatives temporarily rejected, if $[T_n(x)]^{1/2} > \frac{3}{4}M_n$ everywhere, while M_n is still $\geqq 2M$, then (8) holds everywhere, and

$$\gamma_n \geqq 2\pi(\tfrac{1}{4}M_n)^2 \geqq 2\pi(\tfrac{1}{2}M)^2 = M^2\pi/2,$$

which again contradicts (7) from a certain point on. So the hypothesis that $M_n \geqq 2M$ can not be sustained for more than a finite number of values of n, which means that *the sums* $T_n(x)$ *are uniformly bounded.*

Let M', independent of n, be a common upper bound for the expressions $f(x)+[T_n(x)]^{1/2}$. Let

$$R_n(x) = f(x) - [T_n(x)]^{1/2},$$

and let μ_n be the maximum of $|R_n(x)|$. Then

$$|[f(x)]^2 - T_n(x)| \leqq M'\mu_n.$$

The maximum of the absolute value of the derivative of $[f(x)]^2$ has already been denoted by λ'. So the lemma of §2 yields the information that

$$(9) \qquad |T_n'(x)| \leqq nM'\mu_n + C\lambda'.$$

The minimum of $f(x)$ is $a>0$. It is certain that $[T_n(x)]^{1/2}$ is not everywhere less than a, for any specified value of n, since the substitution of the constant a^2 for $T_n(x)$ would then give the integral a smaller value than the alleged minimum γ_n. If $[T_n(x)]^{1/2}$ has a minimum less than a, it must take on the value a and all values between a and the minimum. Suppose for the moment that the minimum is less than $\frac{1}{2}a$. Let y_1 be a point where $[T_n(x)]^{1/2} = \frac{3}{4}a$, and y_2 a point at which $[T_n(x)]^{1/2}=\frac{1}{2}a$, y_1 and y_2 being adjacent points of their respective categories, so that neither of the values $\frac{1}{2}a$, $\frac{3}{4}a$ is taken on anywhere between them. Since $[T_n(x)]^{1/2}$ is uniformly bounded, μ_n is bounded, and it follows from (9) that $|T_n'(x)|$ has an upper bound of the order of magnitude of n. Between y_1 and y_2, where $[T_n(x)]^{1/2}$ is never less than $\frac{1}{2}a$, the derivative of $[T_n(x)]^{1/2}$ is likewise less than a constant multiple of n in absolute value, say

$$\left|\frac{d}{dx}[T_n(x)]^{1/2}\right| < bn.$$

Hence

$$\tfrac{1}{4}a = [T_n(y_1)]^{1/2} - [T_n(y_2)]^{1/2} < bn|y_1 - y_2|,$$

$$|y_1 - y_2| > a/(4bn).$$

But $f(x)-[T_n(x)]^{1/2} \geqq \frac{1}{4}a$ throughout this interval, since the terms of the difference are respectively not less than a and not greater than $\frac{3}{4}a$. Consequently

$$\gamma_n \geqq a^3/(64bn),$$

which contradicts (7) as soon as n is sufficiently large. The contradiction arises from the hypothesis that the minimum of $[T_n(x)]^{1/2}$ is less than $\frac{1}{2}a$. Apart from a finite number of values of n, which can be left out of account without affecting the question of convergence, it must be that

$$(10) \qquad [T_n(x)]^{1/2} \geqq \tfrac{1}{2}a$$

everywhere. It will be understood henceforth that this is the case.

Taken in conjunction with (10), the relation (9) signifies that

$$\left|\frac{d}{dx}[T_n(x)]^{1/2}\right| \leqq (nM'\mu_n + C\lambda')/a,$$

and if λ denotes the maximum of $|f'(x)|$, as before,

$$|R_n'(x)| \leqq Gn\mu_n + H,$$

where $G=M'/a$, $H=(C\lambda'/a)+\lambda$, both quantities being independent of n.

The rest of the proof runs along familiar lines. If $H>Gn\mu_n$, for a specified value of n, then

$$\mu_n < H/(Gn). \tag{11}$$

If $H \leqq Gn\mu_n$,

$$|R_n'(x)| \leqq 2Gn\mu_n,$$

and $|R_n(x)|$, attaining the value μ_n at some point, remains greater than or equal to $\frac{1}{2}\mu_n$ throughout an interval of length at least $1/(2Gn)$. This implies that $\gamma_n \geqq \mu_n^2/(8Gn)$. But it has been seen in (7) that $\gamma_n \leqq k_1/n^2$, if k_1 is used to denote the quantity $2\pi(K\lambda')^2/a^2$, independent of n. So, under the present hypothesis with regard to the relative magnitudes of H and $Gn\mu_n$,

$$\mu_n \leqq (8Gk_1/n)^{1/2}. \tag{12}$$

For every value of n, possibly with a finite number of exceptions, as previously noted, μ_n is subject to one or the other of the relations (11), (12), and certainly approaches zero as n becomes infinite. *This means that* $[T_n(x)]^{1/2}$ *converges uniformly toward* $f(x)$.

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